diagonals and by calculating the LU factorizations of the coefficient matrices only at selected cross-sectional planes (this is similar to a Newton's chord method). A small amount of under-relaxation was found necessary on the corrections made to the velocities and pressures.

Results

Calculations have been made for the experimental configuration of Humphrey et al.[4] The coordinate system is shown in Fig. 1. The duct is of square cross section ($d=0.04$ m) and has a radius of curvature of 0.072 m. The curved duct is preceded and followed by straight sections. The Reynolds number of the flow based on the duct width and the inlet bulk velocity v_b is 792.0. The flow is considered fully developed at a station 0.2 m (5 d) upstream of the 0 deg of the bend. The present calculations were made with a $(58 \times 15 \times 11)$ (θ,r,z) grid that is nearly the same size as that used by Humphrey et al. $(60 \times 15 \times 10)$. Because of symmetry conditions, only half of the duct was solved.

Fully developed duct flow profiles were first generated by solving the equations of a straight duct. These were then prescribed at the inlet plane. A zero derivative exit boundary condition was prescribed at $x=10$ d in the aft straight duct. The calculations were started with simplistic guesses to the velocity and pressure fields. The normalized maximum residuals in the momentum and continuity equations are monitored with iteration number. The present calculations converge rapidly to residuals of 10^{-3} in about 40 iterations. The CPU time on an IBM 3033 was 8 min, which reflects a factor of 2.5 improvement over the time quoted by Humphrey et al. (after considering the relative speeds of the computers involved). Additional computational efficiencies can be gained by developing a reliable iterative algorithm in place of the direct inversion procedure.

Figure 2 shows the calculated secondary flow patterns at four axial locations. It can be seen that the secondary flow is already present upstream of the 0 deg station as a consequence of the ellipticity in the flow. The secondary velocities increase in magnitude with bend angle, reaching a value in excess of 50% of the bulk velocity in regions close to the bottom wall.

Figure 3 shows the development of the axial velocity at different heights from the bottom wall. It is observed that in the initial region of the bend the flow has separated from the outer sidewall at $z=1$ mm as a consequence of the large adverse pressure gradient. The development of the axial velocity profiles agrees satisfactorily with the experimental data and calculations of Humphrey et al. Further improvement in the agreement is possible, however, through the use of more grid nodes in regions of large-velocity gradients.

References

[1]Pratap, V. S. and Spalding, D. B., "Numerical Computations of the Flow in Curved Ducts," *The Aeronautical Quarterly*, Vol. 26, Pt. 3, Aug. 1975, pp. 219-228.

[2]Ghia, K. N. and Sokhey, J. S., "Laminar Incompressible Viscous Flow in Curved Ducts of Regular Cross Sections," *Transactions of ASME, Journal of Fluids Engineering*, Vol. 99, Dec. 1977, pp. 640-648.

[3]Joseph, B., Smith, E. P., and Adler, R. J., "Numerical Treatment of Laminar Flow in Helically Coiled Tubes of Square Cross Section," *AIChE Journal*, Vol. 21, No. 5, Sept. 1975, pp. 965-974.

[4]Humphrey, J. A. C., Taylor, A. M. K., and Whitelaw, J. H., "Laminar Flow in a Square Duct of Strong Curvature," *Journal of Fluid Mechanics*, Vol. 83, Pt. 3, 1977, pp. 509-527.

[5]Mori, Y., Uchida, Y., and Ukon, J., "Forced Convective Heat Transfer in a Curved Channel with a Square Cross Section," *International Journal of Heat and Mass Transfer*, Vol. 14, 1971, pp. 1787-1805.

[6]Pratap, V. S., "Flow and Heat Transfer in Curved Ducts," Ph.D. Thesis, University of London, London, 1975.

[7]Rowe, M., "Measurements and Computations of Flow in Pipe Bends," *Journal of Fluid Mechanics*, Vol. 43, 1970, pp. 771-783.

[8]Berger, S. A., Talbot, L., and Yao, L-S., "Flow in Curved Pipes," *Annual Review of Fluid Mechanics*, Vol. 15, 1983, pp. 461-512.

[9]Patankar, S. V. and Spalding, D. B., "A Calculation Procedure for Heat, Mass, and Momentum Transfer in Three-dimensional Parabolic Flows," *International Journal of Heat and Mass Transfer*, Vol. 15, 1972, pp. 1787-1806.

[10]Pratap, V. S. and Spalding, D. B., "Fluid Flow and Heat Transfer in Three-dimensional Duct Flows," *International Journal of Heat and Mass Transfer*, Vol. 19, 1976, pp. 1183-1193.

[11]Eisenstat, S. C., Gursky, M. C., Schultz, M. H., and Sherman, A. H., "Yale Sparse Matrix Package: The Nonsymmetric Codes," Yale University, New Haven, CT, Res. Rept. 114, 1975.

Unsteady Transport Effects on Diffusion Flame Stability

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Nomenclature

$a_1,a_2,a_3,a_4,$
b_1,b_2,b_3,b_4 = complex constants
c_1,c_2 = complex constants having "nontransport" and "transport" properties, respectively
L_1,L_2 = complex constants
Remaining nomenclature is identical to that of Ref. 1.

Introduction

ANALYTICAL studies of steady-state diffusion flames are generally based on the Burke-Schumann formulation.[2] If the Burke-Schumann geometry is slightly modified to permit fuel/oxidizer flows up to large radial distances, the resulting diffusion flames induce a steady-state velocity profile that tends to be constant near the axis (because of axisymmetry) and at large radial distances (because of the diminishing effect of the diffusion flames). In order to study the dynamic stability of these flames, it is necessary to generalize the formulation to the unsteady case and to carry out a classical linearized stability analysis, taking care to handle correctly the conditions[3] at the unsteady flame. Such an analysis leads to an eigenvalue differential equation whose solution for the eigenvalue yields the growth (or decay) rate of the disturbances along with their frequency. In such analyses, it is usual, as a first approximation, to neglect unsteady transport effects as these give rise to higher-order derivative terms in the stability formulation. Such simplification generally results in some loss of information concerning the stability problem and the resulting mathematical formulation can be referred to as a simplified eigenvalue problem. In the context of the Burke-Schumann problem, the unsteady transport effects refer to unsteady mass and energy diffusion and unsteady energy conduction. Neglecting these effects corresponds to the formal limit Peclet number $P_c \rightarrow \infty$.

An analysis along these lines can be found in Ref. 1, whose problem geometry and linearized unsteady equations constitute a convenient starting point for the present work. In the above analysis, the loss of information referred to earlier manifests itself as an inability of the problem formulation to distinguish between self-excited and damped disturbances. Mathematically, the eigenvalue differential

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equation is of order two instead of four. Instability is predicted, with heat release being the most important cause of instability.

The present work will distinguish between self-excited and damped disturbances by dealing with unsteady transport effects in a limited way. The objective is to extract the needed information without resorting to a full numerical solution of the problem.

Stability Formulation with Unsteady Transport Effects

Equations (19-22) and (24) of Ref. 1, with $n=0$, along with the following more general higher-order replacement for Eq. (23), and the condition that the disturbance amplitude functions $f_x(r)$ remain bounded at $r=0$ and as $r\to\infty$, constitute the stability formulation with unsteady transport effects included,

$$-\bar{\rho}\bar{D}\frac{\mathrm{d}^2 f_y}{\mathrm{d}r^2}-\frac{\bar{\rho}\bar{D}}{r}\frac{\mathrm{d}f_y}{\mathrm{d}r}+f_y(\bar{\rho}\bar{w}\alpha_z-\bar{\rho}\bar{D}\alpha_z^2+\bar{\rho}\alpha_t)+\bar{\rho}\bar{y}\frac{\mathrm{d}f_u}{\mathrm{d}r}$$
$$+f_u\frac{1}{r}\frac{\partial}{\partial r}(\bar{\rho}\bar{y}r)-\bar{D}\frac{\partial\bar{y}}{\partial r}\frac{\mathrm{d}f_\rho}{\mathrm{d}r}+f_\rho\left[-\frac{\partial}{\partial r}\left(\bar{D}\frac{\partial\bar{y}}{\partial r}\right)+\bar{y}\bar{w}\alpha_z\right.$$
$$\left.+\frac{\partial}{\partial z}(\bar{y}\bar{w})-\frac{\bar{D}}{r}\frac{\partial\bar{y}}{\partial r}+\alpha_t\bar{y}\right]+f_w\left[\bar{\rho}\bar{y}\alpha_z+\frac{\partial}{\partial z}(\bar{\rho}\bar{y})\right]$$
$$-\left\{\sigma(\alpha_t+\bar{w}\alpha_z)f_p\bigg/\left[\sum_{j=1}^{N}h_j^{(0)}W_j(\nu_j'-\nu_j'')\right]\right\}=0 \tag{1}$$

Since y represents two different variables (with different values for σ in each case), the above equation represents two equations, the first a linear combination of the oxidizer and fuel species conservation equations and the second a linear combination of the energy and fuel species conservation equations. Only the very weakest of the transport-related terms have been neglected in Eq. (1). The other equations referred to above are those of continuity, momentum, and state. All of the above are in their linearized unsteady form with Fourier components being assigned to the disturbances. The restriction $n=0$ implies that only axisymmetric disturbances are being considered.

As mentioned in the Introduction, the steady-state diffusion flames induce a steady-state velocity profile $\bar{w}$ that tends to be constant near the axis ($\xi=0$) and at large radial distances (large ξ). In such regions, it is possible to combine the above equations into a single fourth-order differential equation for the pressure amplitude function f_p. The result is

$$\frac{\mathrm{d}^4 f_p}{\mathrm{d}\xi^4}+\frac{2}{\xi}\frac{\mathrm{d}^3 f_p}{\mathrm{d}\xi^3}-\frac{1}{\xi^2}\frac{\mathrm{d}^2 f_p}{\mathrm{d}\xi^2}+\frac{1}{\xi^3}\frac{\mathrm{d}f_p}{\mathrm{d}\xi}$$
$$+L_1\left(\frac{\mathrm{d}^2 f_p}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}f_p}{\mathrm{d}\xi}-\frac{L_2}{L_1}f_p\right)=0 \tag{2}$$

where

$$L_1=-\left[(G+VF)^2\frac{M}{V}+P_c\left(\frac{G}{V}+F\right)-2F^2\right]$$

$$L_2=(G+VF)^2\frac{F^2M}{V}-(G+VF)^2\left(\frac{G}{V}+F\right)\frac{P_cM}{V\gamma}$$
$$+P_cF^2\left(\frac{G}{V}+F\right)-F^4$$

It is very important to note that Eq. (2) is valid only in regions where the nondimensional steady-state velocity V is essentially a constant and in each such region V has to be assigned its value in that region. It is appropriate at this stage to introduce the nontransport and transport constants c_1 and c_2, respectively, as

$$c_1^2,c_2^2=(L_1\pm\sqrt{L_1^2+4L_2})/2 \tag{3}$$

where the plus sign is associated with c_1^2 and the negative sign with c_2^2.

It can be shown that Eq. (2) admits Bessel function solutions in terms of $c_1\xi$ and $c_2\xi$, thus forming a system of four linearly independent solutions. Under the very weak assumption that the quotient of Mach number squared and Peclet number is small compared to unity, i.e., $M/P_c\ll 1$, simplified expressions for c_1^2 and c_2^2 can be obtained as

$$c_1^2\simeq F^2-\frac{M}{V\gamma}(G+VF)^2+\frac{M^2}{4P_cV}(G+VF)^3$$

$$c_2^2\simeq-\frac{P_c}{V}(G+VF)+F^2-\frac{(\gamma-1)}{\gamma}\frac{M}{V}(G+VF)^2$$
$$-\frac{M^2}{4P_cV}(G+VF)^3$$

c_1^2 is easily identified as the nontransport parameter obtained in Ref. 1, implying that the two solutions for the amplitude function f_p obtained from the second-order eigenvalue differential equation can also be recovered from the fourth-order equation in the limit $P_c\to\infty$, at least in regions where the steady-state velocity profile is a constant.

Validity of the Simplified Eigenvalue Problem

A necessary condition for the validity of the simplified eigenvalue problem is that the eigenfunctions of the exact problem reduce everywhere to those of the simplified problem in the limit $P_c\to\infty$ after boundary conditions have been applied in each case. This implies that the transport-related solutions involving $c_2\xi$ must have no part to play in this limit. In particular, this requirement must also be satisfied in regions where the steady-state velocity profile is a constant, i.e., near $\xi=0$ and for large ξ. Bearing this in mind, self-excited and damped disturbances are considered in both of the regions.

Self-Excited Disturbances (Region around $\xi=0$)

In the region near $\xi=0$, it is convenient to write the general solution of Eq. (2) in the form

$$f_p=a_1J_0(c_1\xi)+a_2Y_0(c_1\xi)+a_3J_0(c_2\xi)+a_4Y_0(c_2\xi)$$

If f_p and its derivatives are to be bounded at $\xi=0$, it is necessary that $a_2=a_4=0$. Thus,

$$f_p=a_1J_0(c_1\xi)+a_3J_0(c_2\xi)$$

If any point $\xi\neq 0$ in this region is considered and if $P_c\to\infty$, the second function above, i.e., $J_0(c_2\xi)$, will increase indefinitely in magnitude unless c_2 is real. It can be shown (by considering the real and imaginary parts of c_2^2) that for self-excited disturbances (i.e., $G_r>0$) c_2 cannot be real as $P_c\to\infty$. Hence, if f_p has to be bounded, it is necessary that $a_3=0$. Thus, $f_p=a_1J_0(c_1\xi)$. For self-excited disturbances, the eigenfunctions of the exact problem reduce to those of the simplified problem in the limit $P_c\to\infty$, in this region.

Damped Disturbances (Region around $\xi=0$)

For damped disturbances the above argument changes, because c_2 can be real as $P_c\to\infty$, provided the frequency of the oscillations is given by $G_i+V^0F_i=0$.† Thus, as $P_c\to\infty$,

†These frequencies are obtained precisely in the simplified eigenvalue problem (Fig. 3 of Ref. 1) for values of the heat release parameter R lying between 0 and about 30. For larger values of R, an increasing inaccuracy develops, which is a consequence of the numerical method used.

$c_2\xi \to \infty$ along the real line and the function $J_0(c_2\xi) \to 0$. f_p remains bounded as $P_c \to \infty$ and the eigenfunctions once again reduce to those of the simplified problem, but with the important qualitative difference that the constant a_3 is no longer zero. This difference is crucial, as will be seen in the following argument. As $P_c \to \infty$, one can consider the limit $\xi \to 0$ in such a way that $c_2\xi$ is either zero or has a finite value. In this limit, the term $a_3 J_0(c_2\xi)$ makes a finite contribution to f_p and the eigenfunctions of the exact problem no longer reduce to those of the simplified problem. The implication is that damped disturbances of the simplified eigenvalue problem are not true asymptotic representations of damped disturbances of the exact problem in the limit $P_c \to \infty$. Physically, these arguments imply that, for damped disturbances, there exists a region around $\xi = 0$ of dimension $1/P_c^{1/2}$ where the effects of unsteady mass and energy diffusion and unsteady energy conduction cannot be neglected. For damped disturbances, these effects have a finite contribution even as $P_c \to \infty$ and the "transport region" reduces to a point.

Self-Excited and Damped Disturbances (Large ξ)

In this region it is convenient to write the general solution of Eq. (2) in the form

$$f_p = b_1 H_0^{(1)}(c_1\xi) + b_2 H_0^{(2)}(c_1\xi) + b_3 H_0^{(1)}(c_2\xi) + b_4 H_0^{(2)}(c_2\xi)$$

The requirement that f_p be bounded as $\xi \to \infty$ will imply that any two of the four constants above be zero. The exact choice depends on the arguments of c_1 and c_2. Equation (3) shows that c_1 and c_2 can each be chosen so as to lie above or below the real axis. In particular, c_1 and c_2 can always be chosen so as to lie above the real axis and f_p can be written without loss of generality as

$$f_p = b_1 H_0^{(1)}(c_1\xi) + b_3 H_0^{(1)}(c_2\xi)$$

This form of f_p satisfies the boundary conditions as $\xi \to \infty$. If now the limit $p_c \to \infty$ is considered, $c_2\xi$ will be infinitely large and $H_0^{(1)}(c_2\xi) \to 0$. Thus, for both self-excited and damped oscillations, the eigenfunctions of the exact problem reduce to those of the simplified problem in the limit $P_c \to \infty$ in this region.

Conclusion

All of the plots presented in Ref. 1 can be considered as valid representations in the limit $P_c \to \infty$, provided G_r is interpreted as having only the positive values, i.e., only self-excited disturbances. The simplified formulation can also represent neutral disturbances provided this is obtained as a limit of the self-excited case. No information on damped disturbances can be obtained from the simplified problem.

The full solution of the exact eigenvalue problem for finite values of P_c can be obtained by a straightforward (but very time-consuming) integration of the differential equations using asymptotic solutions developed here for initialization. The eigenvalue and relationships between the arbitrary constants can be obtained by a process of iteration from the matching of the numerical solutions. Such an exercise is not considered worthwhile by the author for two reasons: the needed physical information has already been extracted, and it would be inappropriate to make the stability analysis highly refined when the basic steady-state solution has some mutually contradictory assumptions.[2] A fruitful line of effort in this area would be the inclusion of fluid mechanical nonlinearities and viscosity effects in solutions of the steady-state diffusion flame problem. It is the author's opinion that chemical kinetic nonlinearities are not important in this problem and that the flame surface approximation can be continued to be used in both the steady and unsteady cases.

References

[1]Sheshadri, T. S., "Stability Analysis of a Class of Diffusion Flames," *AIAA Journal*, Vol. 23, Jan. 1985, pp. 88-94.

[2]Williams, F. A., *Combustion Theory*, Addison-Wesley Publishing Co., Reading, MA, 1965, pp. 37-44.

[3]Sheshadri, T. S., "Linearized Unsteady Flame Surface Approximation Result in Complex Notation," *AIAA Journal*, Vol. 21, Dec. 1983, pp. 1770-1772.

The Role of Damping on the Stability of Short Beck's Columns

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Introduction

THE Bernoulli-Euler theory, which is used extensively in the analysis of dynamic systems that can be approximated by beams, neglects the important effects of deformation due to shear and rotatory inertia. Inclusion of these effects complicated the partial differential equation governing the dynamics of the beam and, thus, a great deal of effort is needed to solve these equations.

The stability of short Beck and Leipholz columns on elastic foundations was studied by Sundararamaiah and Venkateswara Rao[1] using the finite element method.

In the present Note we will confirm the results of Ref. 1, which indicate that for a Timoshenko beam resting on an elastic foundation subjected to a follower force, the elastic foundation has a destabilizing effect on the beam. The effects of viscoelastic and viscous damping on the stability of columns resting on an elastic foundation subjected to a follower force coupled with the effect of shear deformation and rotatory inertia will be examined.

Differential Equations and Boundary Conditions

The coupled equations for the total deflection y and the bending slope ψ for a cantilevered beam subjected to a follower force P are given as (see Ref. 2)

$$I\left(E + E^*\frac{\partial}{\partial t}\right)\frac{\partial^2\psi}{\partial x^2} + sGA\left(\frac{\partial y}{\partial x} - \psi\right) - \rho I\frac{\partial^2\psi}{\partial t^2} = 0$$

$$\rho A\frac{\partial^2 y}{\partial t^2} - sGA\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial\psi}{\partial x}\right) + P\frac{\partial^2 y}{\partial x^2} + Ky + c\frac{\partial y}{\partial t} = 0 \quad (1)$$

where

E = modulus of elasticity
I = area moment of inertia of cross section
E^* = coefficient of internal dissipation, assumed to be viscoelastic of the Voigt-Kelvin type
G = modulus of rigidity
A = cross-sectional area
s = numerical shape factor for cross section
ρ = density
K = constant elastic foundation modulus
c = viscous damping coefficient

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